Geometry of Quantum Mechanics in complex projective spaces

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Geometrization of Quantum Mechanics

Describing quantum systems in terms of **geometric structures.**

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Why?

- Standard formulation of Quantum Mechanics presents a mathematical structure that is linear and algebraic (operators in Hilbert spaces)
- Classical Mechanics can be mathematically formulated in a broad and elegant differential geometric framework (symplectic manifolds, Hamiltonian fields, Poisson structures...).

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Phylosophical goal: A unified quantum/classical geometric scenario!

Technical goal: Application of powerful geometric tools that are well-known in Classical Mechanics to quantum problems.

Geometrization of Quantum Mechanics

Some landmarks

- **T. W. B. Kibble** *Geometrization of quantum mechanics*, Comm. Math. Phys. 65 (1979)
- **A. Ashtekar and T. A. Schilling** *Geometry of quantum mechanics*, AIP Conf. Proc. 342 (1995)
- **D.C. Brody and L.P. Hughston** *Geometric quantum mechanics*, J. Geom. Phys. 38 (2001)
- **J. Clemente-Gallardo and G. Marmo** *Basics of quantum mechanics, geometrization and some applications to quantum information*, Int. J. Geom. Methods Mod. Phys. 5(6) (2008)

Summary

Geometric Hamiltonian formulation of QM

- Quantum Mechanics in a classical-like fashion
- From operators to phase space functions

Geometry and quantum control

- Notions of quantum controllability
- Differential geometry and quantum controllability

Classical tools

Phase space

A classical system with n spatial degrees of freedom is described in a $2n$-**dimensional symplectic manifold** $(\mathcal{M}, \omega)$.

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Dynamics

A curve in $(a, b) \ni t \mapsto x(t) \in \mathcal{M}$ satisfying Hamilton equations:

$$\frac{dx}{dt} = X_H(x(t))$$

$H : \mathcal{M} \to \mathbb{R}$ is the *Hamiltonian function*.
X_H is the *Hamiltonian vector field*, given by: $\omega_x(X_H, \cdot) = dH_x(\cdot)$

Classical tools

Statistical description

The state is a $\mathcal{C}^1$-function ρ on $\mathcal{M}$ and dynamics is described by the Liouville equation

$$\frac{\partial \rho}{\partial t} + \{\rho, H\}_{PB} = 0$$

Poisson bracket: $\{f, g\}_{PB} := \omega(X_f, X_g)$.

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Physical quantities are real smooth function on $\mathcal{M}$:
The Observable C^*-algebra is:

$$\mathcal{A} = \mathcal{C}^\infty(\mathcal{M})$$

Classical expecation value of $f : \mathcal{M} \to \mathbb{R}$ on ρ:

$$\langle f \rangle_\rho = \int_\mathcal{M} f(x)\rho(x)d\mu(x)$$

QM in a classical-like fashion

Standard formulation of QM in a Hilbert space $\mathcal{H}$:

Quantum states: $D(\mathcal{H}) = \{\sigma \in \mathfrak{B}_1(\mathcal{H}) | \sigma \geq 0, tr(\sigma) = 1\}$
Quantum observables: Self-adjoint operators in $\mathcal{H}$.

Pure states (extreme points of the convex set D) are in bijective correspondence with projective rays in $\mathcal{H}$:

$$\mathcal{P}(\mathcal{H}) = \frac{\mathcal{H}}{\sim} \qquad \psi \sim \phi \;\Leftrightarrow \exists \alpha \in \mathbb{C} \setminus \{0\} \; s.t. \; \psi = \alpha\phi$$

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$\dim\mathcal{H} = n < +\infty$
$\mathcal{P}(\mathcal{H})$ is a real $(2n-2)$-dimensional manifold with the following characterization of tangent space:

$p \in \mathcal{P}(\mathcal{H})$: $\forall v \in T_p\mathcal{P}(\mathcal{H}) \; \exists A_v \in \mathfrak{H}(\mathcal{H})$ s.t. $v = -i[A_v, p]$.

$\mathfrak{H}(\mathcal{H})$ is the space of hermitian operators on $\mathcal{H}$.

$\mathcal{P}(\mathcal{H})$ as a Kähler manifold

Symplectic form: $\omega_p(u, v) := -i\, k\, tr([A_u, A_v]p) \qquad k > 0$

Riemannian metric:

$$g_p(u, v) := -k\, tr(([A_u, p][A_v, p] + [A_v, p][A_u, p])p) \qquad k > 0$$

Almost complex form: $j_p : T_p\mathcal{P}(\mathcal{H}) \ni v \mapsto i[v, p] \in T_p\mathcal{P}(\mathcal{H})$
$p \mapsto j_p$ is smooth and $j_p j_p = -id$ for any $p \in \mathcal{P}(\mathcal{H})$:

$$\omega_p(u, v) = g_p(u, j_p v)$$

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Quantum observables as phase space functions

$$\mathcal{O} : \mathfrak{H}(\mathcal{H}) \ni A \mapsto f_A : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$$

Equivalence Hamilton/Schrödinger dynamics:

$$\frac{dp}{dt} = -i[H, p(t)] \quad \Leftrightarrow \quad \frac{dp}{dt} = X_{f_H}(p(t))$$

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Quantum states as Liouville densities

$$\mathcal{S} : D(\mathcal{H}) \ni \sigma \mapsto \rho_\sigma : \mathcal{P}(\mathcal{H}) \to [0, 1]$$

Equivalence quantum/classical expectation values:

$$\langle A \rangle_\rho = \mathrm{tr}(A\sigma) = \int_{\mathcal{M}} f_A(p)\rho_\sigma(p)d\mu(p)$$

From operators to functions

Definition

A map $f : \mathcal{P}(\mathcal{H}) \to \mathbb{C}$ is called **frame function** if there is $W_f \in \mathbb{C}$ s.t.

$$\sum_{p \in N} f(p) = W_f$$

for any $N \subset \mathcal{P}(\mathcal{H})$ s.t. $d_g(p_1, p_2) = \frac{\pi}{2}$ for $p_1, p_2 \in \mathcal{P}(\mathcal{H})$ with $p_1 \neq p_2$ and N is maximal w.r.t. this property.

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$$\mathcal{F}^2(\mathcal{H}) := \{f : \mathcal{P}(\mathcal{H}) \to \mathbb{C} |\, f \in \mathcal{L}^2(\mathcal{P}(\mathcal{H}), \mu),\ f \text{ is a frame function}\}$$

Theorem (V. Moretti, D.P. 2014)

Phase space functions describing quantum observables are real functions in $\mathcal{F}^2(\mathcal{H})$ and obtained from operators by:

$$\mathcal{O} : \mathfrak{H}(\mathcal{H}) \ni A \mapsto f_A \qquad f_A(p) = k\, tr(Ap) + \frac{1-k}{n} tr(A) \quad k > 0$$

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Theorem (Ashtekar et al. 1995)

A vector field X on $\mathcal{P}(\mathcal{H})$ is the Hamiltonian vector field of a quantum observable (i.e. $X(p) = -i[A, p]$ with $A \in \mathfrak{H}(\mathcal{H})$) if and only if

$$\mathcal{L}_X g = 0$$

C*-algebra of quantum observables in terms of functions

$\mathcal{O} : \mathfrak{H}(\mathcal{H}) \ni A \mapsto f_A \quad$ – linear extension → $\quad \mathcal{O} : \mathfrak{B}(\mathcal{H}) \to \mathcal{F}^2(\mathcal{H})$

$\mathcal{F}^2(\mathcal{H})$ as C*-algebra of observables

-) Involution: $A = \mathcal{O}(f)$, $A^* = \mathcal{O}(\overline{f})$;
-) $\star$ - product: $f \star g = \mathcal{O}\left(\mathcal{O}^{-1}(f)\mathcal{O}^{-1}(g)\right)$:

$$f \star h = \frac{i}{2}\{f, h\}_{PB} + \frac{1}{2}G(df, dh) + f \cdot h \qquad k = 1$$

-) Norm: $|||f||| = \| \mathcal{O}^{-1}(f) \|$

$$|||f||| = \frac{1}{k}\left\| f - \frac{1-k}{n}\int_{\mathcal{P}(\mathcal{H})} f \, d\mu \right\|_{\infty} \qquad k > 0$$

where $d\mu$ is the volume form induced by g.

Quantum control

Controlled n-level quantum system

$$i\hbar\frac{d}{dt}|\psi\rangle = \left[H_0 + \sum_{i=1}^{m} H_i u_i(t)\right]|\psi(t)\rangle \qquad (*)$$

with initial condition $|\psi(0)\rangle = |\psi_0\rangle$.

Pure state controllability

The n-level system is **pure state controllable** if for every pair $|\psi_0\rangle, |\psi_1\rangle \in \mathcal{H}$ there exixst controls $u_1, \ldots, u_m$ and $T > 0$ such that the solution $|\psi\rangle$ of $(*)$ satisfies

$$|\psi(T)\rangle = |\psi_1\rangle$$

.

Quantum control

Controlled n-level quantum system

$$i\hbar \frac{d}{dt} U(t) = \left[H_0 + \sum_{i=1}^{m} H_i u_i(t) \right] U(t) \qquad (**)$$

with initial condition $U(0) = \mathbb{I}$.

Complete controllability

The n-level system is **complete controllable** if for any unitary operator $U_f \in U(n)$ there exist controls $u_1, \ldots, u_n$ and $T > 0$ such that the solution U of $(**)$ satisfies

$$U(T) = U_f$$

Differential geometry and quantum controllability

Geometric Hamiltonian formulation

$$\dot{p}(t) = X_0(p(t)) + \sum_{i=1}^{m} X_i(p(t))u_i(t)$$

X_i are the Hamiltonian fields on $\mathcal{P}(\mathcal{H})$ defined by the classical-like Hamiltonians obtained with our prescription.

Accessibility algebra

The smallest Lie subalgebra $\mathcal{C}$ of the Lie algebra of smooth vector fields on $\mathcal{P}(\mathcal{H})$ containing the fields $X_0, \ldots, X_m$.

Accessibility distribution

$$\mathcal{C}(p) := \text{span}\{X(p) \mid X \in \mathcal{C}\}$$

Theorem (D.P. 2016)

A quantum system is pure state controllable if and only if the following condition is satisfied:

$$T_p\mathcal{P}(\mathcal{H}) = span\{X(p)|X \in \mathcal{C}\}$$

for some $p \in \mathcal{P}(\mathcal{H})$.

The proof is based on this proposition:

$$A \in \mathcal{L} \iff X_{f_{-iA}} \in \mathcal{C}$$

where $\mathcal{L}$ is the Lie algebra generated by $-iH_0, ..., -iH_1$.

Corollary

A quantum system is completely controllable if and only if

$$\mathcal{C} = \mathfrak{Kill}(\mathcal{P}(\mathcal{H}))$$

An example

Consider a controlled 4-level quantum system whose dynamical Lie algebra $\mathcal{L}$ is given by the matrices of the form:

$$A = \begin{pmatrix} -ia & c & z & d \\ e & ib & f & w \\ -\overline{z} & d & ia & e \\ f & -\overline{w} & c & -ib \end{pmatrix},$$

where $a, b, c, d, e, f \in \mathbb{R}$ and $z, w \in \mathbb{C}$.

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where $a, b, c, d, e, f \in \mathbb{R}$ and $z, w \in \mathbb{C}$.
Let $p = diag(1, 0, 0, 0)$ and calculate:

$$X_A(p) = \begin{pmatrix} 0 & -c & -z & -d \\ e & 0 & 0 & 0 \\ -\bar{z} & 0 & 0 & 0 \\ f & 0 & 0 & 0 \end{pmatrix},$$

$\dim \mathcal{C}(p) = 6 = \dim T_p\mathcal{P}(\mathcal{H})$. Pure state controllability!

References

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Thank you for your attention!