Gravitational interaction through a feedback mechanism

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We study the models of Kafri, Taylor and Milburn (KTM) and Tilloy and Diósi (TD), that implement gravity between quantum systems through a continuous measurement and feedback mechanism. We extend the KTM model to a three-dimensional scenario for an arbitrary number of particles, and show that it cannot be considered as a linearized approximation of the TD model. We consider the most natural scenarios for the implementation of a full Newtonian gravitational interaction, and argue that the TD model is the only physically consistent one among these scenarios.

I. INTRODUCTION

Gravity is astonishingly well described by the classical theory of General Relativity as an effect of the space-time deformation due to the presence of a mass [1]. Its unique description renders it difficult to be promoted to a quantum framework. Indeed, the standard quantization techniques lead to non-renormalizable theories [2]. This implies that these theories are not fundamental, and that they should be accounted only as effective theories [3]. Moreover, it may suggest that gravity should be treated as classical at a fundamental level [4, 5], thus implying that its quantization is not necessary.

An open point of dealing with a purely classical framework of gravity is how to treat the case of two quantum systems or the hybrid case of a quantum system interacting gravitationally with a classical one. Some proposals for hybrid dynamics were presented [4, 6], but they lead to inconsistencies, as for example the violation of the correspondence principle [7]. To deal with these inconsistencies, one can phenomenologically include a classical noise, which would lead to the loss of the deterministic trajectory of the classical system under the influence of the quantum one [6, 8–10]. In such a way, one can construct a model inducing a classical behaviour at the macroscopic level, which is mediated by effective forces between mass densities [5, 11]. An example of the latter class of models is that of Kafri, Taylor and Milburn (KTM) studying the gravitational interaction among two point-like masses [12], as well as the model proposed by Tilloy and Diósi (TD) for the study of a generic mass density [13].

In this work, we study the KTM and the TD models, and describe their similarities, differences and issues. We extend the KTM model for N particles in a three dimensional space. We analyse the requirements for the necessary regularization mechanism of the TD model and explicitly derive its conditions. In particular, we construct a family of smearing functions for the case of local operations and classical communication (LOCC) dynamics, which is described below. Then, we compare both models finding that they predict qualitatively different decoherence effects, and thus the TD model cannot be considered as the extension of the KTM model. We also discuss the most natural scenarios where a full Newtonian interaction model is implemented through a continuous measurement and feedback framework, and argue that among them the TD model is the only one which is physically consistent.

II. KAFRI-TAYLOR-MILBURN MODEL

In the KTM model [12], one considers gravity as a classical interaction and implements it through a two-step mechanism. The first step is a weak continuous measurement [14] of the positions $\hat{x}$ of the masses constituting the system. Then, the outcome of the measurement is fed to the other masses through a classical communication channel to effectively implement the gravitational interaction [15, 16]. This second step corresponds to the implementation of a feedback dynamics. Now, since the measurement of the positions of the masses has an intrinsic error, the evolution of the system will be characterized by unavoidable noisy dynamics. Thus, this two-step mechanism leads to a decoherence mechanism along-side the desired effective gravitational attraction between different masses [6].

To be quantitative, KTM consider a system composed of two masses m_1 and m_2, which are harmonically suspended at an initial distance d, as shown in Figure 1, and coupled through gravity. Since there are only two masses, the problem can be fully studied in one dimension. Assuming that the harmonic trapping is sufficiently strong and thus the fluctuations of the masses are small with respect to d, one can Taylor expand the gravitational interaction up to the second order in the relative displacement. Then, with a suitable choice of coordinates, the Hamiltonian of the system reads $\hat{H} = \hat{H}_0 + \hat{H}_{\rm grav}$, where $\hat{H}_0 = \sum_{\alpha=1}^{2} \hat{p}_\alpha^2/2m_\alpha + \frac{1}{2} m_\alpha \Omega_\alpha^2 \hat{x}_\alpha^2$ is the Hamiltonian of a pair of harmonic oscillators, while $\hat{H}_{\rm grav}$ describes the linearized interaction due to gravity:

$$\hat{H}_{\rm grav} = K \hat{x}_1 \hat{x}_2, \tag{1}$$

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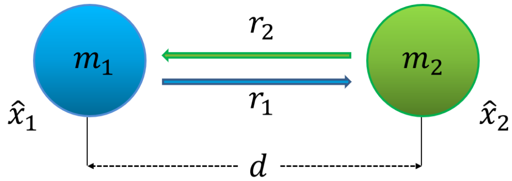

FIG. 1: Graphical representation of the KTM model. Two particles are placed at a initial distance d with respect to each other. Their positions are measured and the corresponding measurement records r_1 and r_2 are used to implement a classical gravitational interaction through a feedback evolution.

where $K = 2Gm_1m_2/d^3$, with G denoting the gravitational constant. Now, the KTM model substitutes the action of $\hat{H}_{\text{grav}}$ with the two-step mechanism above described: *i)* measurement of the positions and *ii)* implementation of the feedback dynamics.

i) Measurement of the positions. – The first step implemented by the KTM model is a weak continuous measurement of the positions of the masses. According to the standard formalism [14], the variation of the state $|\psi_\text{m}\rangle$ due to such a measurement is given by

$$\begin{aligned}\mathrm{d}\,|\psi_\text{m}\rangle = \sum_{\alpha=1}^{2}\Big(&-\frac{\gamma_\alpha}{8\hbar^2}(\hat{x}_\alpha-\langle\hat{x}_\alpha\rangle)^2\mathrm{d}t\\ &+\frac{\sqrt{\gamma_\alpha}}{2\hbar}(\hat{x}_\alpha-\langle\hat{x}_\alpha\rangle)\mathrm{d}W_{\alpha,t}\Big)\,|\psi_\text{m}\rangle,\end{aligned}\tag{2}$$

where $\langle\hat{x}_\alpha\rangle = \langle\psi_\text{m}|\,\hat{x}_\alpha\,|\psi_\text{m}\rangle$ and the two noises $W_{\alpha,t}$ are standard independent Wiener processes. The parameters γ_α denote the information rate gained by the measurement.

ii) Implementation of the feedback dynamics. – After the continuous measurement of the positions of the masses, the system undergoes the feedback evolution. This is implemented by replacing $\hat{H}_{\text{grav}}$ with the feedback Hamiltonian $\hat{H}_{\text{fb}}$, which reads

$$\hat{H}_{\text{fb}} = \chi_{12}r_1\hat{x}_2 + \chi_{21}r_2\hat{x}_1,\tag{3}$$

with χ_{12} and χ_{21} denoting real constants yet to be determined. The key element describing the KTM model is the measurement record r_α, which encodes the classical information about the position of the α-th particle [14]:

$$r_\alpha = \langle\hat{x}_\alpha\rangle + \frac{\hbar}{\sqrt{\gamma_\alpha}}\frac{\mathrm{d}W_{\alpha,t}}{\mathrm{d}t}.\tag{4}$$

The measurement record r_α is a random variable, centered at the expectation value $\langle\hat{x}_\alpha\rangle$ and with a variance defined by the Wiener process $W_{\alpha,t}$ and the information gain rate γ_α. We describe in Appendix A the construction of the measurement record, its identification as a stochastic quantity, and the description of the evolution of a quantum state subject to a continuous measurement.

The Hamiltonian $\hat{H}_{\text{fb}}$ leads to a feedback evolution for the state, which is given by

$$\mathrm{d}\,|\psi_{\text{fb}}\rangle = -\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{2}\left[\frac{i}{\hbar}r_\alpha+\frac{\chi_{\alpha\beta}\hat{x}_\beta}{2\gamma_\alpha}\right]\chi_{\alpha\beta}\hat{x}_\beta\mathrm{d}t\,|\psi_{\text{fb}}\rangle.\tag{5}$$

We report its derivation in Appendix B. The full dynamics of the state $|\psi\rangle$ is given by the combining the contributions in Eq. (2) and Eq. (5). This reads

$$\begin{aligned}\mathrm{d}\,|\psi\rangle = \Bigg\{&-\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{2}\left[\frac{i}{\hbar}r_\alpha+\frac{\chi_{\alpha\beta}\hat{x}_\beta}{2\gamma_\alpha}\right]\chi_{\alpha\beta}\hat{x}_\beta\mathrm{d}t\\ &+\sum_{\alpha=1}^{2}\left[-\frac{\gamma_\alpha}{8\hbar^2}(\hat{x}_\alpha-\langle\hat{x}_\alpha\rangle)^2\mathrm{d}t+\frac{\sqrt{\gamma_\alpha}}{2\hbar}(\hat{x}_\alpha-\langle\hat{x}_\alpha\rangle)\mathrm{d}W_{\alpha,t}\right]\\ &-\frac{i}{2\hbar}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{2}\chi_{\alpha\beta}\hat{x}_\beta(\hat{x}_\alpha-\langle\hat{x}_\alpha\rangle)\mathrm{d}t\Bigg\}\,|\psi\rangle\,,\end{aligned}\tag{6}$$

where the first line corresponds to the feedback contribution, the second line to that of the continuous measurement, while the third line is the Itô term arising from the combined effect of the continuous measurement and feedback contributions. The corresponding master equation for the density operator can be obtained by performing the stochastic average $\mathbb{E}[\cdot]$ over the Wiener processes, and it reads

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = &-\frac{i}{\hbar}\left[\hat{H}_0,\hat{\rho}\right]-\frac{i}{2\hbar}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{2}\chi_{\alpha\beta}\left[\hat{x}_\beta,\{\hat{x}_\alpha,\hat{\rho}\}\right]\\ &-\sum_{\alpha=1}^{2}\left(\frac{\gamma_\alpha}{8\hbar^2}+\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{2}\frac{\chi_{\beta\alpha}^2}{2\gamma_\beta}\right)[\hat{x}_\alpha,[\hat{x}_\alpha,\hat{\rho}]],\end{aligned}\tag{7}$$

where $\hat{\rho} = \mathbb{E}[|\psi\rangle\langle\psi|]$ and we also included the free evolution described by $\hat{H}_0$. The master equation is the sum of two contributions, in addition to the dynamical evolution due to $\hat{H}_0$). The first one is given by the last term of the first line in Eq. (7) and will mimic the gravitational interaction among the two masses. The second one, which is constituted by the second line in Eq. (7), describes the noise part of the dynamics and leads to decoherence. To correctly mimic the gravitational interaction one sets $\chi_{12} = \chi_{21} = K$, and the master equation becomes

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = &-\frac{i}{\hbar}\left[\hat{H}_0+K\hat{x}_1\hat{x}_2,\hat{\rho}\right]\\ &-\sum_{\alpha=1}^{2}\left(\frac{\gamma_\alpha}{8\hbar^2}+\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{2}\frac{K^2}{2\gamma_\beta}\right)[\hat{x}_\alpha,[\hat{x}_\alpha,\hat{\rho}]]\,.\end{aligned}\tag{8}$$

Hence, one recovers the quantum gravitational interaction defined by $\hat{H}_{\text{grav}}$ in Eq. (1). Therefore, the KTM prescription effectively retrieves the quantum gravitational interaction through a classical communication channel. However, one needs to pay the price of having extra decoherence effects, whose strength is determined by the parameters γ_α. In the particular case of two equal masses $m_1 = m_2$, it is reasonable to assume that the measurement process has the same rate, thus we set $\gamma = \gamma_1 = \gamma_2$. While an estimate of the value of γ can be determined only through experiments, one can theoretically quantify the value of γ that minimizes the decoherence effects. The particular structure of the decoherence terms in Eq. (8) allows to perform such a minimization. Accordingly, the second line of Eq. (8) becomes

$$-\frac{K}{2\hbar}\sum_{\alpha=1}^{2}[\hat{x}_\alpha,[\hat{x}_\alpha,\hat{\rho}]], \tag{9}$$

and corresponds to an information gain rate equal to $\gamma_{\min} = 2\hbar K$ [12].

In summary, the KTM model implements a local operation and classical communication (LOCC) dynamics [17, 18], where the local operation is provided by the continuous measurement of local variables while the feedback dynamics works as a classical communication [19]. Such a LOCC dynamics simulates the action of a quantum gravitational field [20], paying the price of having a decoherence mechanism affecting the system dynamics.

III. LINEARIZED-GRAVITY GENERALIZATION OF THE KTM MODEL

The KTM model describes the gravitational interaction of two particles and provides a master equation [cf. Eq. (7)] which is valid in one dimension. The result of KTM was extended by Altamirano *et al.* in Refs. [21, 22]. Their model describes the gravitational interaction between two bodies of N_1 and N_2 constituents, respectively. Due to the symmetry of the problem [21], Altamirano *et al.* can effectively reduce the dynamics to one dimension. The corresponding master equation is given by

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = &-\frac{i}{\hbar}\left[\hat{H}_0 + \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N_1+N_2}\hat{V}_{\alpha\beta},\hat{\rho}\right]\\ &-\frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N_1+N_2}\Gamma_{\alpha\beta}\left([\hat{x}_\alpha,[\hat{x}_\alpha,\hat{\rho}]] + [\hat{x}_\beta,[\hat{x}_\beta,\hat{\rho}]]\right),\end{aligned} \tag{10}$$

where $\hat{V}_{\alpha\beta}$ is the gravitational potential expanded up to second order in the positions $\hat{x}_\alpha$, and the constants $\Gamma_{\alpha\beta}$ are the decoherence rates.

We now generalize the KTM model to three dimensions, following their original approach [12]. The Taylor expansion up to the second order in position operators of the gravitational interaction between two particles of mass m_α and m_β reads

$$\hat{V}(\hat{\mathbf{x}}_\alpha,\hat{\mathbf{x}}_\beta) \approx \sum_{\alpha=1}^{N}\hat{Y}_\alpha + \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\sum_{l,j=1}^{3}K_{\alpha\beta lj}\hat{x}_{\alpha l}\hat{x}_{\beta j} \tag{11}$$

where the Greek letters α,β denote the particles and the Latin letters l,j denote the directions in space. The single particle operator $\hat{Y}_\alpha$ is a second-order polynomial in the components of the position operator $\hat{\mathbf{x}}_\alpha$, while the constants $K_{\alpha\beta lj}$ are defined as:

$$K_{\alpha\beta lj} = Gm_\alpha m_\beta\left[\frac{3d_{\alpha\beta l}d_{\alpha\beta j}}{|\mathbf{d}_{\alpha\beta}|^5} - \frac{\delta_{lj}}{|\mathbf{d}_{\alpha\beta}|^3}\right], \tag{12}$$

where the vector $\mathbf{d}_{\alpha\beta}$ joins the positions of the two masses. This is the generalization of K introduced in Eq. (1).

We now apply the two-step mechanism for implementing the gravitational interaction. The variation of the wavefunction due to the continuous measurements of the positions $\hat{x}_{\alpha l}$ is described by:

$$\begin{aligned}\mathrm{d}\left|\psi_{\mathrm{m}}\right\rangle = \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\sum_{l,j=1}^{3}\Big(&-\frac{\gamma_{\alpha\beta lj}}{8\hbar^2}\left(\hat{x}_{\alpha l} - \langle\hat{x}_{\alpha l}\rangle\right)^2\mathrm{d}t\\ &+\frac{\gamma_{\alpha\beta lj}}{2\hbar}\left(\hat{x}_{\alpha l} - \langle\hat{x}_{\alpha l}\rangle\right)\mathrm{d}W_{\alpha\beta lj,t}\Big)\left|\psi_{\mathrm{m}}\right\rangle,\end{aligned} \tag{13}$$

where the parameters $\gamma_{\alpha\beta lj}$ are the information rates gained by the measurements, and the noises $\mathrm{d}W_{\alpha\beta lj,t}$ are standard independent Wiener processes in each of the four indices α,β,l,j. We notice that there are $3N$ operators $\hat{x}_{\alpha l}$, one for each value of α and l. The corresponding measurement records read

$$r_{\alpha\beta lj} = \langle\hat{x}_{\alpha l}\rangle + \frac{\hbar}{\sqrt{\gamma_{\alpha\beta lj}}}\frac{\mathrm{d}W_{\alpha\beta lj,t}}{\mathrm{d}t}, \tag{14}$$

and they are represented graphically in Fig. 2. The measurement record $r_{\alpha\beta lj}$ embeds the information about the position of the particle α along the l-th direction, which influences the dynamics of the particle β along the j-th direction. The explicit dependence of the measurement records on both the particles and its components allows to recover the gravitational interaction as expressed in Eq. (11). It is important to remark that, while there are no restrictions on the direction indices, the indices α and β are not allowed to coincide. This implies that the particles will not obtain information about themselves. Therefore, the total number of measurement records is $9N(N-1)$.

For a fixed value of α, the measurement record $r_{\alpha\beta lj}$ corresponding to the particle of mass m_α contributes to the evolution of the positions of the other masses through the following feedback Hamiltonian

$$\hat{H}_{\mathrm{fb}} = \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\sum_{l,j=1}^{3}\chi_{\alpha\beta lj}r_{\alpha\beta lj}\hat{x}_{\beta j}, \tag{15}$$

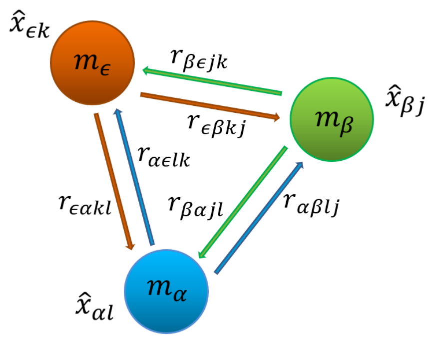

FIG. 2: Graphical representation of the KTM model scheme for $N = 3$ particles in three dimensions. In general, for each particle α and each dimension l, one defines $3 \times (N-1)$ measurement records. This allows to implement a consistent feedback evolution between the N particles in three dimensions.

where we assume the following symmetry for the constants $\chi_{\alpha\beta lj} = \chi_{\beta\alpha lj} = \chi_{\alpha\beta jl}$. The corresponding feedback evolution for the wavefunction is given by

$$\mathrm{d}\left|\psi_{\mathrm{fb}}\right\rangle = -\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \sum_{l,j=1}^{3} \left[\frac{i}{\hbar} r_{\alpha\beta lj} + \frac{\chi_{\alpha\beta lj}}{2\gamma_{\alpha\beta lj}}\right] \chi_{\alpha\beta lj} \hat{x}_{\beta j} \mathrm{d}t \left|\psi_{\mathrm{fb}}\right\rangle . \tag{16}$$

Following the procedure drawn above and reported in Appendix A and Appendix B, we arrive at the corresponding master equation

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} =& -\frac{i}{\hbar}\left[\hat{H}_0, \hat{\rho}\right] - \frac{i}{2\hbar} \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \sum_{l,j=1}^{3} K_{\alpha\beta lj} \left[\hat{x}_{\alpha l}\hat{x}_{\beta j}, \hat{\rho}\right] \\ & - \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \sum_{l,j=1}^{3} \left(\frac{\gamma_{\alpha\beta lj}}{8\hbar^2} + \frac{1}{2}\frac{K^2_{\alpha\beta lj}}{\gamma_{\alpha\beta lj}}\right) \left[\hat{x}_{\alpha l}, \left[\hat{x}_{\alpha l}, \hat{\rho}\right]\right] ,\end{aligned} \tag{17}$$

where we absorbed the operators $\hat{Y}_\alpha$ in the Hamiltonian $\hat{H}_0$. The master equation (17) is the three-dimensional generalization of that of the KTM model. This generalization, as well as the model by Altamirano *et al.*, consider a pairwise interaction between the particles that constitute the system. The unitary evolution describes the gravitational interaction of the constituents of the system. However, it is modified by the terms in the second line of Eq. (17), which lead to decoherence effects. We notice that the decoherence terms in the above equation contain only the position operators of the same particle along the same direction. Such decoherence terms can be suitably minimized by fixing an appropriate value of the information rates $\gamma_{\alpha\beta lj}$. Thus, as for the KTM model, one obtains a minimum decoherence coefficient that can be eventually tested experimentally.

Finally, we comment about another possible implementation of gravity for N particles through a feedback mechanism in one dimension, which we study in Appendix D. The key difference with the generalization leading to Eq. (17) is in the definition of the measurement record corresponding to $\hat{x}_\alpha$. In this alternative scenario, one defines a single measurement record r_α, as in Eq. (4), that is fed to all the other particles. In contrast,$r_{\alpha\beta lj}$, which is defined in Eq. (14), is used in the dynamical evolution of particle β only. This leads to inconsistencies at the level of the master equation, which are further discussed in Appendix D.

IV. TILLOY-DIÓSI MODEL

The Tilloy-Diósi (TD) model [13, 23] follows the KTM idea of implementing the gravitational interaction through a feedback evolution for arbitrary distances. Instead of the measurement of positions of the masses, the TD model implements the continuous measurement of the mass density of the system. This choice allows a straightforward extension to the case of identical particles, where one expresses the mass density operator as a mass-weighted sum over different species of particles. In this setting, Eq. (1) is replaced by

$$\hat{H}_{\mathrm{grav}} = \frac{1}{2}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathcal{V}(\mathbf{x}-\mathbf{y})\hat{\mu}(\mathbf{x})\hat{\mu}(\mathbf{y}), \tag{18}$$

where $\mathcal{V}(\mathbf{x}-\mathbf{y}) = -G/|\mathbf{x}-\mathbf{y}|$ is the gravitational potential and $\hat{\mu}(\mathbf{x})$ is the mass density operator of the system. It is important to notice that $\hat{H}_{\mathrm{grav}}$ contains standard divergences due to self-interactions. However, we underline that this does not differ from the standard classical Newtonian gravity, and can be similarly treated.

Similarly to the KTM model, TD replace $\hat{H}_{\mathrm{grav}}$ with a two-step process mimicking a gravitational interaction: the measurement of $\hat{\mu}(\mathbf{x})$ and the implementation of the corresponding feedback dynamics. The variation of the wavefunction due to the continuous measurement of the mass density is given by

$$\begin{aligned}\mathrm{d}\left|\psi_{\mathrm{m}}\right\rangle =& \Big[-\frac{1}{8\hbar^2}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\gamma(\mathbf{x},\mathbf{y})\left(\hat{\mu}(\mathbf{x}) - \langle\hat{\mu}(\mathbf{x})\rangle\right) \\ & \times\left(\hat{\mu}(\mathbf{y}) - \langle\hat{\mu}(\mathbf{y})\rangle\right)\mathrm{d}t \\ & + \frac{1}{2\hbar}\int \mathrm{d}\mathbf{x}\left(\hat{\mu}(\mathbf{x}) - \langle\hat{\mu}(\mathbf{x})\rangle\right)\delta\mu_t(\mathbf{x})\mathrm{d}t\Big]\left|\psi_{\mathrm{m}}\right\rangle.\end{aligned} \tag{19}$$

This is the analogue of Eq. (2) in the KTM model. Here, we introduced $\langle\hat{\mu}(\mathbf{x})\rangle = \langle\psi_{\mathrm{m}}|\,\hat{\mu}(\mathbf{x})\,|\psi_{\mathrm{m}}\rangle$ and the noise $\delta\mu_t(\mathbf{x})$, which corresponds to $\frac{\mathrm{d}W_t}{\mathrm{d}t}$ of the KTM model and is characterized by

$$\begin{aligned}\mathbb{E}[\delta\mu(\mathbf{x})] &= 0, \\ \mathbb{E}\left[\delta\mu_t(\mathbf{x})\delta\mu_{t'}(\mathbf{y})\right] &= \gamma(\mathbf{x},\mathbf{y})\delta(t-t'),\end{aligned} \tag{20}$$

where $\gamma(\mathbf{x},\mathbf{y})$ describes the spatial correlation of the noise. The latter is assumed to be symmetric, thus satisfying $\gamma(\mathbf{x},\mathbf{y}) = \gamma(\mathbf{y},\mathbf{x})$.

In analogy with Eq. (3), we introduce the feedback Hamiltonian

$$\hat{H}_{\rm fb} = \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathcal{V}(\mathbf{x}-\mathbf{y})\hat{\mu}(\mathbf{x})\mu(\mathbf{y}), \tag{21}$$

where $\mu(\mathbf{y})$ is the measurement record of the mass density corresponding to the measurement process in Eq. (19). For each point in space, the measurement record is defined as

$$\mu(\mathbf{x}) = \langle\hat{\mu}(\mathbf{x})\rangle + \hbar\int \mathrm{d}\mathbf{y}\,\gamma^{-1}(\mathbf{x},\mathbf{y})\delta\mu_t(\mathbf{y}). \tag{22}$$

Here, $\gamma^{-1}(\mathbf{x},\mathbf{y})$ is the inverse function of $\gamma(\mathbf{x},\mathbf{y})$, for which the following relation holds

$$(\gamma\circ\gamma^{-1})(\mathbf{x},\mathbf{y}) = \int \mathrm{d}\mathbf{r}\,\gamma(\mathbf{x},\mathbf{r})\gamma^{-1}(\mathbf{r},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}). \tag{23}$$

We report a method to construct the inverse kernel $\gamma^{-1}(\mathbf{x},\mathbf{y})$ in Appendix C. The corresponding feedback wavefunction dynamics is given by [23]

$$\begin{aligned}\mathrm{d}\left|\psi_{\rm fb}\right\rangle = -\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\,\Big\{&\frac{i}{\hbar}\mathcal{V}(\mathbf{x}-\mathbf{y})\mu(\mathbf{y})\\ &+\frac{1}{2}(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V})(\mathbf{x},\mathbf{y})\hat{\mu}(\mathbf{y})\Big\}\,\hat{\mu}(\mathbf{x})\mathrm{d}t|\psi_{\rm fb}\rangle.\end{aligned} \tag{24}$$

By merging the action of Eq. (19) and Eq. (24), we obtain the full evolution of the wavefunction, which reads

$$\begin{aligned}\mathrm{d}\left|\psi\right\rangle = \Big(&-\frac{i}{\hbar}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathcal{V}(\mathbf{x}-\mathbf{y})\hat{\mu}(\mathbf{x})\mu(\mathbf{y})\mathrm{d}t\\ &-\frac{1}{2}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\left(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V}\right)(\mathbf{x},\mathbf{y})\hat{\mu}(\mathbf{x})\hat{\mu}(\mathbf{y})\mathrm{d}t\\ &-\frac{1}{8\hbar^2}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\gamma(\mathbf{x},\mathbf{y})\left(\hat{\mu}(\mathbf{x})-\langle\hat{\mu}(\mathbf{x})\rangle\right)\left(\hat{\mu}(\mathbf{y})-\langle\hat{\mu}(\mathbf{y})\rangle\right)\mathrm{d}t\\ &+\frac{1}{2\hbar}\int \mathrm{d}\mathbf{x}\left(\hat{\mu}(\mathbf{x})-\langle\hat{\mu}(\mathbf{x})\rangle\right)\delta\mu_t(\mathbf{x})\mathrm{d}t\\ &-\frac{i}{2\hbar}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathcal{V}(\mathbf{x}-\mathbf{y})\hat{\mu}(\mathbf{x})\left(\hat{\mu}(\mathbf{y})-\langle\hat{\mu}(\mathbf{y})\rangle\right)\mathrm{d}t\Big)\left|\psi\right\rangle.\end{aligned} \tag{25}$$

As in Eq. (6), such a dynamical equation now includes the feedback and the continuous measurement contributions, as well as the Itô term resulting from the combination of the two steps. The corresponding master equation reads

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} =& -\frac{i}{\hbar}\left[\hat{H}_0+\hat{H}_{\rm grav},\hat{\rho}\right]\\ &-\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}D(\mathbf{x},\mathbf{y})\left[\hat{\mu}(\mathbf{x}),\left[\hat{\mu}(\mathbf{y}),\hat{\rho}\right]\right],\end{aligned} \tag{26}$$

where we added the free Hamiltonian $\hat{H}_0$ and defined

$$D(\mathbf{x},\mathbf{y}) = \left[\frac{\gamma}{8\hbar^2}+\frac{1}{2}\left(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V}\right)\right](\mathbf{x},\mathbf{y}), \tag{27}$$

which is the decoherence kernel of the model. Similarly to the KTM model, TD retrieve the quantum gravitational interaction, whose unitary evolution is modified by the decoherence due to the measurement and the feedback dynamics.

The advantages of the TD model over the KTM model are two. First, one considers the full form of the gravitational potential and not only its Taylor expansion near an equilibrium position. Second, the use of mass density operator allows to study also identical particles.

V. THE DIVERGENCES IN THE TD MODEL

At this level, the decoherence term of the master equation (26) is only formal. In particular, one still needs to fix the form of the noise kernel $\gamma(\mathbf{x},\mathbf{y})$, which is arbitrary. In the following we show that, under the assumption that $\gamma(\mathbf{x},\mathbf{y})$ is invariant under translations, i.e. $\gamma(\mathbf{x},\mathbf{y}) = \gamma(\mathbf{x}-\mathbf{y})$, any choice of γ leads to divergences in the decoherence term of the master equation. To do so, let us consider a system of point-like particles, whose mass density is given by

$$\hat{\mu}(\mathbf{x}) = \sum_{\alpha=1}^{N} m_\alpha\delta(\mathbf{x}-\hat{\mathbf{x}}_\alpha). \tag{28}$$

By substituting it, once expressed in terms of its Fourier transform, in the decoherence term of Eq. (26), we obtain

$$\begin{aligned}&\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}D(\mathbf{x},\mathbf{y})\left[\hat{\mu}(\mathbf{x}),\left[\hat{\mu}(\mathbf{y}),\hat{\rho}\right]\right]\\ &=\sum_{\alpha,\beta=1}^{N} m_\alpha m_\beta\int\frac{\mathrm{d}\mathbf{k}\tilde{D}(\mathbf{k})}{(2\pi\hbar)^{3/2}}\left[e^{-\frac{i}{\hbar}\mathbf{k}\cdot\hat{\mathbf{x}}_\alpha},\left[e^{\frac{i}{\hbar}\mathbf{k}\cdot\hat{\mathbf{x}}_\beta},\hat{\rho}\right]\right],\end{aligned} \tag{29}$$

where $\tilde{D}(\mathbf{k})$ is the Fourier transform of $D(\mathbf{x}-\mathbf{y})$, which inherits the translational invariance from γ. We can identify the divergences in the TD model by studying the terms in the above sum corresponding to the same particle, which are those with $\alpha=\beta$. These are proportional to

$$\int \mathrm{d}\mathbf{k}\,\tilde{D}(\mathbf{k})\left(2\hat{\rho}-e^{-\frac{i}{\hbar}\mathbf{k}\cdot\hat{\mathbf{x}}_\alpha}\hat{\rho}e^{\frac{i}{\hbar}\mathbf{k}\cdot\hat{\mathbf{x}}_\alpha}-e^{\frac{i}{\hbar}\mathbf{k}\cdot\hat{\mathbf{x}}_\alpha}\hat{\rho}e^{-\frac{i}{\hbar}\mathbf{k}\cdot\hat{\mathbf{x}}_\alpha}\right). \tag{30}$$

One finds that the first term $\int \mathrm{d}\mathbf{k}\tilde{D}(\mathbf{k})$ diverges. Indeed, straightforward calculations show that according to Eq. (27):

$$\int \mathrm{d}\mathbf{k}\,\tilde{D}(\mathbf{k}) = \int \mathrm{d}\mathbf{k}\left(\frac{\tilde{\gamma}(\mathbf{k})}{8\hbar^2}+8\pi^2\hbar^4G^2\frac{\widetilde{\gamma^{-1}}(\mathbf{k})}{k^4}\right), \tag{31}$$

where the Fourier transform of the inverse of the noise kernel $\widetilde{\gamma^{-1}}(\mathbf{k})$ is related to $\tilde{\gamma}(\mathbf{k})$ due to Eq. (23):

$$\tilde{\gamma}(\mathbf{k})\widetilde{\gamma^{-1}}(\mathbf{k}) = \frac{1}{(2\pi\hbar)^3}, \tag{32}$$

Then, Eq. (31) can be written in terms of $\tilde{\gamma}(\mathbf{k})$ as

$$\int \mathrm{d}\mathbf{k}\, \tilde{D}(\mathbf{k}) = \int \mathrm{d}\mathbf{k} \left(\frac{\tilde{\gamma}(\mathbf{k})}{8\hbar^2} + \frac{\hbar G^2}{\pi} \frac{1}{k^4 \tilde{\gamma}(\mathbf{k})} \right). \tag{33}$$

Equation (33) is the sum of two contributions: the continuous measurement, which gives the first term, and the application of the gravitational interaction through a feedback evolution, which provides the second term. Before analyzing the general case, let us study two particular correlation kernels.

The first case corresponds to LOCC dynamics, which requires that the dynamics acts only locally [13]. A noise correlation function reflecting this property is proportional to a Dirac-delta. Thus, we set

$$\gamma(\mathbf{x}-\mathbf{y}) = A\delta(\mathbf{x}-\mathbf{y}), \tag{34}$$

where A is an arbitrary constant. In such a case, we have that $\tilde{\gamma}(\mathbf{k}) = A/(2\pi\hbar)^{3/2}$. By substituting the latter expression in Eq. (33), one gets that none of its contributions is convergent. Thus, in the TD model, the assumptions of having point-like particles and implementing a LOCC dynamics lead to divergences.

As second case of interest, we consider a Gaussian correlation kernel $\gamma(\mathbf{z}) = (2\pi\sigma^2)^{-3/2} \exp\left[-\mathbf{z}^2/(2\sigma^2)\right]$. In this case, one have $\tilde{\gamma}(\mathbf{k}) = (2\pi\hbar)^{-3/2} \exp(-\mathbf{k}^2\sigma^2/2\hbar^2)$. Now, by substituting the latter expression in Eq. (33), we find that although the continuous measurement contribution converges, the feedback contribution is still divergent.

Next, we show the general case: any choice of $\gamma(\mathbf{x}-\mathbf{y})$ leads to divergences. Similarly to what was done in the KTM model, we minimize the decoherence kernel $\tilde{D}(\mathbf{k})$ with respect to $\tilde{\gamma}(\mathbf{k})$. The minimum is reached for $\tilde{\gamma}(\mathbf{k}) = G(2\pi\hbar)^{3/2}/(\pi^2 k^2)$, which corresponds to $\gamma(\mathbf{x}-\mathbf{y}) = -2\hbar\mathcal{V}(\mathbf{x}-\mathbf{y})$. Such a correlation kernel leads to the decoherence rate of the Diósi-Penrose model [8, 9, 13], which is still divergent [24]. Indeed, Eq. (33) reads

$$\int \mathrm{d}\mathbf{k}\tilde{D}(\mathbf{k}) = \frac{2(2\pi\hbar)^{1/2}G}{\hbar} \int_0^\infty \mathrm{d}k \to \infty. \tag{35}$$

Since the latter choice of $\gamma(\mathbf{x}-\mathbf{y})$ provides the minimum decoherence rate, we deduce that Eq. (33), and subsequently the master equation (26), diverges for any choice of $\gamma(\mathbf{x}-\mathbf{y})$. The next section is dedicated to the regularization of the decoherence rate through the use of a smearing function.

VI. REGULARIZATION OF THE TD MODEL

We need a regularization process to avoid divergences in the decoherence terms in Eq. (26). This regularization mechanism is typically also applied to the Diósi-Penrose model [24], by introducing a smearing function. For the TD model, the contributions to the decoherence term are those from the measurement part, through $\gamma(\mathbf{x}-\mathbf{y})$, and from the feedback evolution, through $(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V}^{-1})(\mathbf{x}-\mathbf{y})$. Both these terms must be regularized. Indeed, the regularization of the noise kernel $\gamma(\mathbf{x}-\mathbf{y})$ alone would only give a different noise kernel $\gamma'(\mathbf{x}-\mathbf{y})$, which is not sufficient to avoid the divergence, as proved before. On the other hand, the regularization of the gravitational potential $\mathcal{V}(\mathbf{x},\mathbf{y})$ could remove the divergences in the feedback contribution, but not that due to the measurement, which is independent from the gravitational interaction. We conclude that the regularization mechanism must be performed by smearing both $\gamma(\mathbf{x}-\mathbf{y})$ and $\mathcal{V}(\mathbf{x}-\mathbf{y})$.

An effective regularization procedure consists in smearing the mass density operator as proposed in Refs. [13, 25]. According to this prescription, we substitute the mass density $\hat{\mu}(\mathbf{x})$ with the smeared one

$$\hat{\nu}(\mathbf{r}) = \int \mathrm{d}\mathbf{x}\, g(\mathbf{x}-\mathbf{r})\hat{\mu}(\mathbf{x}), \tag{36}$$

where $g(\mathbf{x}-\mathbf{y})$ is a suitable smearing function. This is equivalent to regularize both the noise kernel $\gamma(\mathbf{x}-\mathbf{y})$ and the gravitational potential $\mathcal{V}(\mathbf{x}-\mathbf{y})$ with the same smearing function. The latter approach is described by the following substitution [13]:

$$\gamma \to g\circ\gamma\circ g, \quad \text{and} \quad \mathcal{V} \to g\circ\mathcal{V}\circ g. \tag{37}$$

In the following, we will implement the smearing of $\gamma(\mathbf{x}-\mathbf{y})$ and $\mathcal{V}(\mathbf{x}-\mathbf{y})$. An appropriate smearing function should remove all the divergences of the master equation (26) for an arbitrary choice of the mass density and of the noise kernel. In particular, $\hat{H}_{\text{grav}}$ in Eq. (18) becomes

$$\hat{H}'_{\text{grav}} = \frac{1}{2}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\; (g\circ\mathcal{V}\circ g)(\mathbf{x}-\mathbf{y})\hat{\mu}(\mathbf{x})\hat{\mu}(\mathbf{y}), \tag{38}$$

and the decoherence kernel defined in Eq. (27) turns into

$$D'(\mathbf{x}-\mathbf{y}) = \left[\frac{g\circ\gamma\circ g}{8\hbar^2} + \frac{1}{2} g\circ\left(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V}\right)\circ g\right](\mathbf{x}-\mathbf{y}). \tag{39}$$

By substituting $\hat{H}_{\text{grav}}$ with $\hat{H}'_{\text{grav}}$ and $D(\mathbf{x}-\mathbf{y})$ with $D'(\mathbf{x}-\mathbf{y})$ in Eq. (26), we obtain

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} =& -\frac{i}{\hbar}\left[\hat{H}_0 + \hat{H}'_{\text{grav}}, \hat{\rho}\right] \\ & - \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} D'(\mathbf{x},\mathbf{y}) \left[\hat{\mu}(\mathbf{x}), \left[\hat{\mu}(\mathbf{y}), \hat{\rho}\right]\right],\end{aligned} \tag{40}$$

which we refer to as the master equation of the Tilloy-Diósi model. To summarize, we are able to retrieve the quantum gravitational interaction paying the price of extra decoherence effect. However, we emphasize that this is not the usual gravitational interaction, but a smeared one.

As a case of interest, we consider the LOCC dynamics, whose noise correlation function is given by Eq. (34). In such a case, the effect of the smearing on the master equation is described in terms of the following functions

$$\begin{aligned}(g\circ\mathcal{V}\circ g)\,(\mathbf{x}-\mathbf{y}) &= -4\pi G\hbar^2\eta_2(\mathbf{x}-\mathbf{y}),\\ (g\circ\gamma\circ g)\,(\mathbf{x}-\mathbf{y}) &= A\eta_0(\mathbf{x}-\mathbf{y}),\\ \left[g\circ\left(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V}\right)\circ g\right](\mathbf{x}-\mathbf{y}) &= \frac{16\pi^2G^2\hbar^4}{A}\eta_4(\mathbf{x}-\mathbf{y}),\end{aligned}\tag{41}$$

where we defined

$$\eta_n(\mathbf{x}-\mathbf{y}) = \int\frac{\mathrm{d}\mathbf{k}}{k^n}\,\tilde{g}^2(\mathbf{k})e^{\frac{i}{\hbar}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})},\tag{42}$$

with $\tilde{g}(\mathbf{k})$ denoting the Fourier transform of $g(\mathbf{x}-\mathbf{y})$. A good smearing function must give finite expressions in Eq. (41), which reflect an appropriate short-distance regularization of the gravitational potential $\mathcal{V}(\mathbf{x}-\mathbf{y})$, the correlation kernel $\gamma(\mathbf{x}-\mathbf{y})$ and the feedback dynamics $\left(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V}\right)(\mathbf{x}-\mathbf{y})$. In turn, one can exploit Eq. (41) to define restrictions for having a well behaving smearing function. In particular, the requirement of the convergence of η_4 prevents the use of some intuitive choices for the smearing. Indeed, if one considers a Gaussian smearing $g(\mathbf{z}) = \left(2\pi\sigma^2\right)^{-3/2}\exp\left(-\mathbf{z}^2/2\sigma^2\right)$, one has that both $\eta_0(\mathbf{x},\mathbf{y})$ and $\eta_2(\mathbf{x},\mathbf{y})$ converge, while $\eta_4(\mathbf{x},\mathbf{y})$, in spherical coordinates, becomes:

$$\eta_4(\mathbf{x}-\mathbf{y}) = \frac{4\pi}{(2\pi\hbar)^3}\int_0^\infty\mathrm{d}k\,\frac{e^{-\sigma^2k^2/\hbar^2}}{k^2}\frac{\sin\left(\frac{k}{\hbar}|\mathbf{x}-\mathbf{y}|\right)}{\frac{k}{\hbar}|\mathbf{x}-\mathbf{y}|},\tag{43}$$

which diverges, since the integrand is not well defined for $k\to 0$.

In the following, we determine the convergence requirements for the coefficients $\eta_n(\mathbf{x},\mathbf{y})$. For the sake of simplicity, we consider only spherical smearing functions, i.e. $\tilde{g}(\mathbf{k}) = \tilde{g}(k)$. In such a case, Eq. (42) simplifies to

$$\eta_n(\mathbf{x}-\mathbf{y}) = 4\pi\int_0^\infty\frac{\mathrm{d}k}{k^{n-2}}\,\tilde{g}^2(k)\frac{\sin\left(\frac{k}{\hbar}|\mathbf{x}-\mathbf{y}|\right)}{\frac{k}{\hbar}|\mathbf{x}-\mathbf{y}|},\tag{44}$$

which converges, for example, for smearing functions of the family $\tilde{g}(k) = k^\beta e^{-\alpha k^2}$ with $\alpha > 0$ and $\beta \geq 1$. Concretely, a smearing function of the form

$$g(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\alpha\hbar)^{7/2}}\left[6\alpha\hbar^2-(\mathbf{x}-\mathbf{y})^2\right]e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{4\alpha\hbar^2}},\tag{45}$$

whose Fourier transform is

$$\tilde{g}(k) = k^2e^{-\alpha k^2},\tag{46}$$

belongs to such a family. In particular, explicit calculations lead to

$$\begin{aligned}\eta_0(\mathbf{z}) &= \frac{\pi^{3/2}}{16\hbar^4(2\alpha)^{11/2}}\left[\mathbf{z}^4+40\alpha\hbar^2\left(6\alpha\hbar^2-\mathbf{z}^2\right)\right]e^{-\frac{\mathbf{z}^2}{8\alpha\hbar^2}},\\ \eta_2(\mathbf{z}) &= -\frac{\pi^{3/2}}{4\hbar^2(2\alpha)^{7/2}}\left(\mathbf{z}^2-12\alpha\hbar^2\right)e^{-\frac{\mathbf{z}^2}{8\alpha\hbar^2}},\\ \eta_4(\mathbf{z}) &= \frac{\pi^{3/2}}{(2\alpha)^{3/2}}e^{-\frac{\mathbf{z}^2}{8\alpha\hbar^2}},\end{aligned}\tag{47}$$

which are well defined also for $|\mathbf{z}| = |\mathbf{x}-\mathbf{y}| \to 0$. Thus, the divergences in the TD model are indeed avoided.

Equation (44) allows to determine a good smearing for the regularization of the TD model for a LOCC dynamics. We remark that this holds only for the particular case of a LOCC dynamics. Indeed, if one considers a different noise kernel $\gamma(\mathbf{x}-\mathbf{y})$, thus releasing the LOCC constraint, one could successfully regularize the dynamics also with more standard choices for the smearing function. For example, if one takes $\gamma(\mathbf{x}-\mathbf{y}) = -2\hbar\mathcal{V}(\mathbf{x}-\mathbf{y})$, a normalized Gaussian smearing of standard deviation σ leads to the following a decoherence kernel

$$D'(\mathbf{x},\mathbf{y}) = \frac{G}{2\hbar|\mathbf{x}-\mathbf{y}|}\,\mathrm{erf}\left(\frac{|\mathbf{x}-\mathbf{y}|}{2\sigma}\right),\tag{48}$$

which behaves well also for $|\mathbf{x}-\mathbf{y}|\to 0$.

VII. COMPARISON BETWEEN THE TD AND KTM MODELS

The TD and KTM models consider the same problem: how to effectively implement the gravitational interaction among two quantum systems by using a continuous measurement and a feedback evolution. The starting point for the TD model is the quantum Hamiltonian in Eq. (18), while for the KTM model, one has its linearized version for point-like particles in Eq. (1). From these, the two models derive the corresponding master equation, which are respectively Eq. (40) (once one implements the regularization procedure) and Eq. (7). In this section, we compare the two models for the case of N particles. For the KTM model, we will consider the generalization of Eq. (17). In addition, we will reduce the TD model to the regime where the gravitational interaction can be treated linearly. By comparing the resulting master equations, we explicitly show that the KTM evolution does not coincide with that of the linearized TD model.

A. Linear limit of the TD model

We now consider the linear regime for the gravitational interaction in the TD model for the case of point-like particles, whose mass density is given by Eq. (28). We show in Fig. 3 a graphical representation of this model for $N = 3$ particles. We can rewrite the position operator of each particle as follows

$$\hat{\mathbf{x}}_\alpha = \mathbf{x}_\alpha^{(0)} + \Delta\hat{\mathbf{x}}_\alpha,\tag{49}$$

where $\Delta\hat{\mathbf{x}}_\alpha$ is the quantum displacement from the initial position $\mathbf{x}_\alpha^{(0)}$. For small displacements, we can approxi-

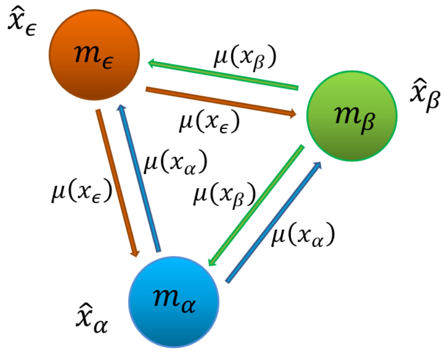

FIG. 3: Graphical representation of the TD model scheme for $N = 3$ particles in three dimensions. To implement the dynamical evolution through the feedback Hamiltonian, each particle receives the same information about the mass density of the constituents.

mate Eq. (40) as

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}_0, \hat{\rho}\right] + \frac{2i\pi G}{\hbar}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\sum_{l,j=1}^{3} m_\alpha m_\beta \eta_{\alpha\beta 2lj}\left[\hat{x}_{\alpha l}\hat{x}_{\beta j}, \hat{\rho}\right] - \sum_{\alpha,\beta=1}^{N}\sum_{l,j=1}^{3} m_\alpha m_\beta \eta_{\alpha\beta lj}\left[\hat{x}_{\alpha l}, \left[\hat{x}_{\beta j}, \hat{\rho}\right]\right], \tag{50}$$

where $\hat{x}_{\alpha l}$ is the component in the l direction of $\Delta\hat{\mathbf{x}}_\alpha$. This choice of notation matches that used in Section II. Moreover, we included the terms coming from $\hat{H}'_{\text{grav}}$ corresponding to the same particle ($\alpha = \beta$) in the definition of $\hat{H}_0$. The parameter $\eta_{\alpha\beta lj}$ is defined as

$$\eta_{\alpha\beta lj} = \left(\frac{\pi^3}{8\hbar^5}\right)^{1/2}\eta_{\alpha\beta 0lj} + (8\pi\hbar)^{1/2}G^2\eta_{\alpha\beta 4lj}, \tag{51}$$

and coefficients $\eta_{\alpha\beta nlj}$ are given by

$$\begin{aligned}
\eta_{\alpha\beta 0lj} &= \int \mathrm{d}\mathbf{k}\, \tilde{g}^2(\mathbf{k})\, \tilde{\gamma}(\mathbf{k}) k_l k_j\, e^{-\frac{i}{\hbar}\mathbf{k}(\mathbf{x}_\alpha^{(0)} - \mathbf{x}_\beta^{(0)})},\\
\eta_{\alpha\beta 2lj} &= \int \frac{\mathrm{d}\mathbf{k}}{k^2}\, \tilde{g}^2(\mathbf{k}) k_l k_j\, e^{-\frac{i}{\hbar}\mathbf{k}\cdot(\mathbf{x}_\alpha^{(0)} - \mathbf{x}_\beta^{(0)})},\\
\eta_{\alpha\beta 4lj} &= \int \frac{\mathrm{d}\mathbf{k}}{k^4}\, \frac{\tilde{g}^2(\mathbf{k})}{\tilde{\gamma}(\mathbf{k})} k_l k_j\, e^{-\frac{i}{\hbar}\mathbf{k}\cdot(\mathbf{x}_\alpha^{(0)} - \mathbf{x}_\beta^{(0)})}.
\end{aligned} \tag{52}$$

To completely determine the coefficients of the linear TD model, one needs only to fix both the smearing function $g(\mathbf{x} - \mathbf{y})$ and the correlation kernel $\gamma(\mathbf{x} - \mathbf{y})$.

B. Linear TD model vs. extended KTM model

We are now able to compare the two models through Eq. (17) and Eq. (50). For both models, the unitary part of the master equations gives the linearized gravitational interaction between different particles. In particular, the dependence from the position operators is the same and the smearing function in Eq. (50) provides some freedom that can be exploited to match the numerical values with Eq. (17).

Conversely, the decoherence terms predicted by the two models have a different functional dependence on the position operators. While the double commutator term in Eq. (17) contains only the position operators corresponding to the *same* particle, the corresponding term in Eq. (50) contains also position operators of *different* particles. Indeed, for $\alpha \neq \beta$, the coefficients $\eta_{\alpha\beta nlj}$ defined in Eq. (52) do not vanish. The presence of these terms in the TD model are due to the use of the same measurement record [cf. Eq. (22)] when constructing the feedback Hamiltonian of Eq. (21). This means that all the constituents of the system are influenced in the same way by the measurement record of the mass density $\mu(\mathbf{y})$ at each point of space.

In contrast, there are $9N(N-1)$ different measurement records [cf. Eq. (14)] in the KTM generalization. In this form, the dynamical evolution of the position of a given particle along one direction is the result of the $3(N-1)$ measurement records coming from the other particles of the system, as shown in Eq. (15). In contrast with the TD model, these measurement records are not the same for each particle.

VIII. FOUR SCENARIOS FOR THE IMPLEMENTATION OF GRAVITY THROUGH A FEEDBACK MECHANISM

The KTM and TD models suggest that, in order to construct a model that implements gravity as a classical channel through a feedback dynamics, it is necessary to consider two fundamental aspects. The first one is the physical observable which is chosen to drive the feedback interaction. Given the nature of the gravitational interaction [cf. Eq. (1) and Eq. (18)], one is constrained to only two reasonable physical choices, namely, the position or the mass density. The KTM model chooses the position, the TD model the mass density. Notice that, in both cases, the observable which is continuously measured also drives the feedback interaction.

The other aspect to take into account is the measurement. In the KTM model and its generalizations, each constituent of the system is measuring the position of the others, thus establishing a pairwise interaction between the constituents of the system. In contrast, in the TD model the mass density of the whole system is measured once at each point of space, and all the constituents use this measurement record in their dynamical evolution.

Here we are interested in the implementation of the full Newtonian interaction $-G/|\mathbf{x}-\mathbf{y}|$, not only its linearized version, within a continuous measurement and feedback framework. We consider a system of point-like particles, as this assumption is enough for the purposes of our discussion. The above remarks lead us to consider four natural scenarios. We can classify them in two main classes, according to the observables that are measured: either the position or the mass density. For each case, one can implement the continuous measurement and feedback as a pairwise interaction between the particles, or through a universal measurement, in which all the constituents use the same information about the other particles in their dynamical evolution. The TD model corresponds to the choice of measuring the mass density of the system, through a universal measurement. We now discuss the other three scenarios.

First, we argue that the two scenarios which consider the measurement of the positions are not viable. The reason is that the Newtonian interaction $\hat{H}_{\text{grav}}$ is inversely proportional to the distance between the constituents of the system. Therefore, if one uses the measurement records r_α of the positions of the particles for the feedback, then $\hat{H}_{\text{grav}}$ [cf. Eq. (18)] will depend nonlinearly on r_α, and therefore one cannot construct a feedback Hamiltonian of the form of Eq. (B1). This is crucial because, without a linear dependence of the feedback on the measurement record, the resulting stochastic equation will not have the standard structure of quantum state diffusion unravelings for completely positive semigroups [26]. This means that the resulting master equation will not be of the Lindblad type, and in general it will not even be closed for the density matrix, thus not representing a satisfactory dynamical evolution of the system. Therefore this option is not viable, without substantial revisions.

Now, let us consider the scenarios in which one measures the mass density. One of the two is the TD model, which implements a universal measurement. The other scenario is that where each particle measures the mass density of the remaining ones. We now show that this second scenario is also inconsistent. Let us consider a system of N point-like particles, where each constituent has a mass density $\hat{\mu}_\alpha(x)$, and let us define the measurement records

$$\mu_{\alpha\beta}(\mathbf{x}) = \langle\hat{\mu}_\alpha(\mathbf{x})\rangle + \hbar\int \mathrm{d}\mathbf{z}\gamma^{-1}_{\alpha\beta}(\mathbf{x},\mathbf{z})\delta\mu_{\alpha\beta,t}(\mathbf{z}), \tag{53}$$

that describe a pairwise measurement of the mass density of each of the constituents of the system. The noises $\delta\mu_{\alpha\beta}(\mathbf{x})$ are characterized by

$$\begin{aligned}\mathbb{E}\left[\delta\mu_{\alpha\beta,t}(\mathbf{x})\right] = 0,\\ \mathbb{E}[\delta\mu_{\alpha\beta,t}(\mathbf{x})\delta\mu_{\alpha'\beta',t'}(\mathbf{y})] = \delta_{\alpha\alpha'}\delta_{\beta\beta'}\gamma_{\alpha\beta}(\mathbf{x},\mathbf{y})\delta(t-t').\end{aligned} \tag{54}$$

The gravitational interaction is implemented through the following feedback Hamiltonian

$$\hat{H}_{\text{fb}} = \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathcal{V}(\mathbf{x},\mathbf{y})\hat{\mu}_\beta(\mathbf{x})\hat{\mu}_{\alpha\beta}(\mathbf{y}). \tag{55}$$

This leads to the following master equation

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} =& -\frac{i}{2\hbar}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathcal{V}'(\mathbf{x},\mathbf{y})\left[\hat{\mu}_\alpha(\mathbf{x})\hat{\mu}_\beta(\mathbf{y}),\hat{\rho}\right]\\ &-\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}D'_{\alpha\beta}(\mathbf{x},\mathbf{y})\left[\hat{\mu}_\alpha(\mathbf{x}),\left[\hat{\mu}_\alpha(\mathbf{y}),\hat{\rho}\right]\right],\end{aligned} \tag{56}$$

where

$$\begin{aligned}\mathcal{V}'(\mathbf{x},\mathbf{y}) = (g\circ\mathcal{V}\circ g)(x,\mathbf{y}),\\ D'_{\alpha\beta}(\mathbf{x},\mathbf{y}) = (g\circ D_{\alpha\beta}\circ g)(\mathbf{x},\mathbf{y}),\end{aligned} \tag{57}$$

with

$$D_{\alpha\beta}(\mathbf{x},\mathbf{y}) = \left[\frac{\gamma_{\alpha\beta}}{8\hbar^2} + \frac{1}{2}\left(\mathcal{V}\circ\gamma^{-1}_{\beta\alpha}\circ\mathcal{V}\right)\right](\mathbf{x},\mathbf{y}). \tag{58}$$

We are interested in the minimization of the decoherence effects which modify the unitary evolution. For this purpose, we take all the correlation kernels $\gamma_{\alpha\beta}(\mathbf{x},\mathbf{y})$ to be equal: $D_{\alpha\beta}(\mathbf{x},\mathbf{y}) = D(\mathbf{x},\mathbf{y})$ for all α,β. Next, one sets $\gamma(\mathbf{x},\mathbf{y}) = -2\hbar\mathcal{V}(\mathbf{x},\mathbf{y})$ as in the TD model, to obtain a minimum for the decoherence kernel. We regularize the model with the Gaussian smearing $g(\mathbf{z}) = (2\pi\sigma^2)^{-3/2}\exp\left(-\mathbf{z}^2/2\sigma^2\right)$.

Let us now consider the interaction between two systems made of $N_1 = 1$ (with mass m_1) and N_2 particles, respectively ($N = N_1 + N_2$). Working in Fourier space, after tracing over the degrees of freedom of the system with N_2 particles we obtain

$$\begin{aligned}&\mathrm{Tr}_{N_2}\left(-\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}D'_{\alpha\beta}(\mathbf{x},\mathbf{y})\left[\hat{\mu}_\alpha(\mathbf{x}),\left[\hat{\mu}_\alpha(\mathbf{y}),\hat{\rho}\right]\right]\right)\\ &=\sum_{\beta=2}^{N}\frac{2Gm_1^2}{4\pi^2\hbar^2}\int\frac{\mathrm{d}\mathbf{k}}{k^2}e^{-\frac{\sigma^2}{\hbar^2}\mathbf{k}^2}\left(e^{\frac{i}{\hbar}\mathbf{k}\cdot\hat{\mathbf{x}}_1}\hat{\rho}_1 e^{-\frac{i}{\hbar}\mathbf{k}\cdot\hat{\mathbf{x}}_1} - \hat{\rho}_1\right)\end{aligned} \tag{59}$$

with $\hat{\rho}_1 = \int \mathrm{d}\mathbf{x}_2\cdots\mathrm{d}\mathbf{x}_N \langle\mathbf{x}_2|\otimes\langle\mathbf{x}_N|\,\hat{\rho}\,|\mathbf{x}_N\rangle\otimes|\mathbf{x}_2\rangle$. Suppose that we work with a delocalized state, such that the second term of Eq. (59) is dominant over the first one. Then, the coherence decays with a rate Γ given by

$$\Gamma = \sum_{\beta=2}^{N}\frac{2Gm_1^2}{4\pi^2\hbar^2}\int\frac{\mathrm{d}\mathbf{k}}{k^2}e^{-\frac{\sigma^2}{\hbar^2}\mathbf{k}^2} = \frac{N_2Gm_1^2}{\sqrt{\pi}\hbar\sigma}. \tag{60}$$

This rate explicitly depends on the second system through its number of constituents. This is an unphysical result, since one can consider for the second system the whole Universe and, therefore, the above result would yield a vastly large decoherence rate. We note that this

Model	System	Observable	no. signals	Correlation	Feedback $\hat{H}_{\text{fb}}$	Master eq.
KTM	2 particles	$\hat{x}_\alpha$ (1D)	2	$\delta_{\alpha\beta}$	$\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{2} \chi_{\alpha\beta} r_\alpha \hat{x}_\beta$	Eq. (8)
KTM (extension)	N particles	$\hat{\mathbf{x}}_\alpha$ (3D)	$9N(N-1)$	$\delta_{\alpha\alpha'}\delta_{\beta\beta'}\delta_{jj'}\delta_{ll'}$	$\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \sum_{l,j=1}^{3} \chi_{\alpha\beta lj} r_{\alpha\beta lj} \hat{x}_{\beta j}$	Eq. (17)
TD (no smearing)	Continuous	$\hat{\mu}(\mathbf{x})$	over $\mathbb{R}^3$	$\gamma(\mathbf{x}-\mathbf{y})$	$\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathcal{V}(\mathbf{x}-\mathbf{y})\hat{\mu}(\mathbf{x})\mu(\mathbf{y})$	Eq. (26)
TD	Continuous	$\hat{\mu}(\mathbf{x})$	over $\mathbb{R}^3$	$(g\circ\gamma\circ g)(\mathbf{x}-\mathbf{y})$	$\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}(g\circ\mathcal{V}\circ g)(\mathbf{x}-\mathbf{y})\hat{\mu}(\mathbf{x})\mu(\mathbf{y})$	Eq. (40)

TABLE I: List of models considered in this work, implementing gravity through a feedback mechanism. For each model we identify the type of *System* (if discrete or continuous), the *Observable* which is continuously measured, and the *Number of signals* produced between the constituents of the system. We report the *Correlations* of the corresponding noises. The *Feedback Hamiltonian* $\hat{H}_{fb}$ and reference to the corresponding *Master equation* are also reported.

inconsistency is not present for the TD model, where under the same scenario the decoherence rate for the first system is:

$$\Gamma_{\text{TD}} = \frac{Gm_1^2}{\sqrt{\pi}\hbar\sigma}, \tag{61}$$

which depends only on the single particle of mass m_1, not on the other system. Therefore, the scenario in which we implement a full gravitational interaction by constructing pairwise measurements of the mass density is not physically consistent (unless, again, one introduces major changes in the construction of the model). We therefore find out that, among the four possible scenarios described above in which one can potentially implement gravity through a continuous measurement and feedback framework, the only physically consistent one is the TD model.

IX. DISCUSSION

The comparison of Eq. (17) and Eq. (50) shows that the KTM model cannot be regarded as the linear approximation of the TD model, once we impose the appropriate correlation kernel [cf. Eq. (34)] and define a smearing function. Indeed, the decoherence effects predicted by the two models are qualitatively different. This is a consequence of the specific implementation of the gravitational interaction in the feedback Hamiltonian $\hat{H}_{\text{fb}}$. We show the differences between the two models in Table I. This allows a classification of the models by considering two main criteria:

i) Gravitational interaction: position vs mass density. – In the KTM model and its generalization the position of the particles is the quantum observables to be continuously measured. Conversely, while in the TD model the whole mass density is measured.
ii) Measurement records: pairwise vs universal. – The implementation of gravity depends crucially also on the number of measurement records (and therefore, the number of noises) used in each of the two models. The KTM and its generalization consider a discrete number of particles, and the measurement records link the particles in pairs: those whose position is measured with those receiving the measurement record. These multiple measurement records lead to decoherence effects in which the double-commutators contain only position operators of a single particle. In contrast, the TD model describes the gravitational interaction also of continuous systems, and there is one measurement record at each point of space, as shown in Table I. When the system is a collection of point-like particles, each of the particles receives the *same* information about the rest of the constituents of the system, not in a pairwise manner as for the KTM model. This leads to decoherence terms in the master equation involving operators belonging to different particles.

In addition, the KTM model does not preserve the symmetry of the wavefunction for systems of identical particles [27]. This represents a further restriction of the applicability of the KTM model when it is compared to the TD model. As mentioned before, the latter allows the study of identical particles, as the mass density operator can be described in the language of second quantization.

We briefly point out that the generalization of the KTM model of Eq. (17) removes by construction the self-interactions between the particles of a system. Conversely, in the TD model such self-interactions are still present, thus not differing from the standard description of gravity for continuous systems.

X. CONCLUSIONS

The main virtue of the KTM [12], and TD [13] models is that the gravitational interaction can be recovered as a classical channel, within a quantum framework. The price to pay are additional decoherence effects which can be minimized but not evaded. In this work, we extended the KTM model to a three-dimensional setting for N particles, without the need of perfoming an effectively one dimensional description of the gravitational interaction between the constituents of a system, as done in Ref. [21].

Secondly, by considering the TD model [13] model for a collection of N point-like particles, we provided a robust analysis of the origin of the divergences of the model. The latter are unavoidable even with the prescription of minimizing the decoherence kernel. Following the regularization mechanism of Ref. [13], we described the conditions that the smearing function must satisfy in order to avoid the divergences in the model. In particular, we showed that smearing functions that work in similar models [24, 25] do not necessarily remove the divergences in the TD model. Moreover, we proposed a family of well-behaved smearing functions that regularize the model under the request of a LOCC dynamics. In particular we found that a LOCC dynamics cannot be implemented for a system of point-like particles without the smearing of both the correlation kernel and the gravitational interaction. Equivalently, we can leave these kernels unaltered, while smearing the mass density of the particles, which effectively become not point-like.

We also showed that the KTM and the TD models provide different predictions for the decoherence effects that appear when implementing gravity through a continuous measurement and feedback framework. Therefore, the KTM model is not an approximated version of the TD model. Although both models are consistent within their range of validity, they describe the gravitational interaction between particles in different ways. The corresponding decoherence terms reflect this fact.

Finally, we discussed the most natural scenarios in which one would implement a continuous measurement and feedback framework for the full Newtonian interaction. We considered two fundamental factors when describing these scenarios, namely, the quantum observable used to construct the feedback Hamiltonian, as well as the type of interaction between the constituents established through the measurement records. We argued that position measurements lead to inconsistent dynamics, and showed that a pairwise measurement of the mass density yields unphysical decoherence rates when studying the gravitational interaction between subsystems. Therefore, we conclude that the TD model is the only physically consistent description of the full Newtonian interaction, within the framework of continuous measurement and feedback so far proposed.

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Appendix A: Continuous measurements

We recall here the main properties characterizing the continuous measurement of an observable $\hat{a}$. The results follow mainly from Refs. [14, 28].

Let us consider the continuous observable $\hat{a}$ with associated eigenstates $\{|a\rangle\}$ satisfying $\hat{a}\,|a\rangle = a\,|a\rangle$. For an infinitesimal time interval Δt, one can construct a parametrized sum of projectors onto the eigenstates of $\hat{a}$

$$\hat{A}(r) = \left(\frac{\gamma\Delta t}{2\pi\hbar^2}\right)^{1/4} \int_{-\infty}^{\infty} \mathrm{d}a \exp\left[-\frac{\gamma\Delta t}{4\hbar^2}\left(\hat{a}-r\right)^2\right] |a\rangle\langle a| \,. \tag{A1}$$

Assuming that the parameter r that characterizes the Gaussian-weighted operators $\hat{A}(r)$ is continuous, one obtains a continuum of measurement results labelled by this parameter r. Denoting by $P(r)$ the probability density of the measurement result r, the mean value $\langle r\rangle$ of r, and the variance σ_r^2 of r are related to those of $\hat{a}$ by

$$\langle r\rangle = \int_{-\infty}^{\infty} rP(r)\mathrm{d}r = \langle\hat{a}\rangle\,, \quad \sigma_r^2 = \langle r^2\rangle - \langle r\rangle^2 = \sigma_{\hat{a}}^2 + \frac{\hbar^2}{\gamma\Delta t}. \tag{A2}$$

Since the time interval Δt is infinitesimal, the probability density $P(r)$ can be approximated as

$$P(r) \approx \frac{1}{\hbar}\sqrt{\frac{\gamma\Delta t}{2\pi}} \exp\left[-\frac{\gamma\Delta t}{2\hbar^2}\left(r-\langle\hat{a}\rangle\right)^2\right]. \tag{A3}$$

From the results of Eq. (A2) and Eq. (A3), r can be written as a stochastic quantity

$$r = \langle\hat{a}\rangle + \frac{\hbar}{\sqrt{\gamma}}\frac{\Delta W_t}{\Delta t}, \tag{A4}$$

where ΔW_t is a Gaussian random variable with zero mean and variance Δt.

By performing a sequence of these measurements, and taking the limit $\Delta t \to 0$, one obtains a so-called continuous measurement, described by

$$r = \langle\hat{a}\rangle + \frac{\hbar}{\sqrt{\gamma}}\frac{\mathrm{d}W_t}{\mathrm{d}t}. \tag{A5}$$

We can see that the measurement records defined in Eq. (4) are just the particular cases of Eq. (A5) in which the observables measured are the position operators $\hat{x}_\alpha$ of the particles, with $\alpha = 1, 2$.

Let us denote by $|\psi\rangle$ the state of a system at a time t before performing a continuous measurement of the observable $\hat{a}$. The evolution of the system will be described by applying the operator $\hat{A}(r)$ to the state $|\psi\rangle$, and performing the limit $\Delta t \to 0$. By demanding that the resulting dynamical equation preserves the norm, one obtains

$$\mathrm{d}\,|\psi\rangle_m = \left\{-\frac{\gamma}{8\hbar^2}\left(\hat{a}-\langle\hat{a}\rangle\right)^2\mathrm{d}t + \frac{\sqrt{\gamma}}{2\hbar}\left(\hat{a}-\langle\hat{a}\rangle\right)\mathrm{d}W_t\right\}|\psi\rangle\,, \tag{A6}$$

so that the result of Eq. (2) is consistent with the general formalism of Eq. (A6). The generalization to a continuous set of observables used in Section IV can be found in Ref. [23].

Appendix B: Feedback evolution

In the Markovian case, the feedback Hamiltonian $\hat{H}_{\rm fb}$ is expressed in terms of the measurement record r of the observable $\hat{a}$ as

$$\hat{H}_{\rm fb} = r\hat{b}, \tag{B1}$$

where $\hat{b}$ is a Hermitian operator. The feedback evolution can be obtained by unitarily evolving the state of the system $|\psi\rangle$ [23]. This gives

$$e^{-\frac{i}{\hbar}\hat{H}_{\rm fb}{\rm d}t}\,|\psi\rangle = |\psi\rangle + {\rm d}\,|\psi\rangle_{\rm fb}\,, \tag{B2}$$

where

$${\rm d}\,|\psi\rangle_{\rm fb} = \left(\left[-\frac{i}{\hbar}\,\langle\hat{a}\rangle\,\hat{b} - \frac{1}{2\gamma}\hat{b}^2\right]{\rm d}t - \frac{i}{\sqrt{\gamma}}\hat{b}\,{\rm d}W_t\right)|\psi\rangle\,. \tag{B3}$$

The full evolution of the system is obtained by considering the contributions of both the continuous measurement of $\hat{a}$ [cf. Eq. (A6)] and the subsequent feedback dynamics driven by $\hat{b}$ as described by Eq. (B3) [14]. Straightforward application of Itô formalism yields the following expression

$${\rm d}\,|\psi\rangle = {\rm d}\,|\psi\rangle_m + {\rm d}\,|\psi\rangle_{\rm fb} + {\rm d}\,|\psi\rangle_{\rm It\hat{o}}\,, \tag{B4}$$

where the last contribution arises from the combined effect of the previous two, and is explicitly given by

$${\rm d}\,|\psi\rangle_{\rm It\hat{o}} = -\frac{i}{2\hbar}\hat{b}\,(\hat{a} - \langle\hat{a}\rangle)\,{\rm d}t\,|\psi\rangle\,. \tag{B5}$$

The total master equation can be derived from Eq. (B4) and is given by

$$\frac{{\rm d}\hat{\rho}}{{\rm d}t} = -\frac{i}{2\hbar}[\hat{b},\{\hat{a},\hat{\rho}\}] - \frac{\gamma}{8\hbar^2}\,[\hat{a},[\hat{a},\hat{\rho}]] - \frac{1}{2\gamma}[\hat{b},[\hat{b},\hat{\rho}]], \tag{B6}$$

where $\hat{\rho} = \mathbb{E}[|\psi\rangle\,\langle\psi|]$ denotes the density operator.

Appendix C: Construction of the correlation kernels

We describe with more detail the relation between a kernel $\mathcal{K}(\mathbf{x}-\mathbf{y})$ and its inverse $\mathcal{K}^{-1}(\mathbf{x}-\mathbf{y})$ by following the approach developed in Ref. [29]. Consider the operator $\mathcal{A}$ which satisfies

$$\mathcal{A}\mathcal{K}(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \tag{C1}$$

where $\mathcal{K}(\mathbf{x}-\mathbf{y})$ is the associated kernel. We define the integral transform

$$u(\mathbf{x}) = \int {\rm d}\mathbf{r}\mathcal{K}(\mathbf{r}-\mathbf{x})f(\mathbf{r}), \tag{C2}$$

and require that the inverse kernel $\mathcal{K}^{-1}(\mathbf{x}-\mathbf{y})$ satisfies

$$\delta(\mathbf{x}-\mathbf{y}) = \int {\rm d}\mathbf{r}\mathcal{K}(\mathbf{x}-\mathbf{r})\mathcal{K}^{-1}(\mathbf{r}-\mathbf{y}). \tag{C3}$$

From these expressions, we can show that

$$f(\mathbf{x}) = \int {\rm d}\mathbf{r}\mathcal{A}\mathcal{K}(\mathbf{r}-\mathbf{x})f(\mathbf{r}), \tag{C4}$$

and equivalently

$$f(\mathbf{x}) = \int {\rm d}\mathbf{r}\mathcal{K}^{-1}(\mathbf{r}-\mathbf{x})u(\mathbf{r}). \tag{C5}$$

The substitution of Eq. (C2) in Eq. (C5) and the comparison with Eq. (C4) lead to

$$\mathcal{K}^{-1}(\mathbf{x}-\mathbf{y}) = \mathcal{A}^2\mathcal{K}(\mathbf{x}-\mathbf{y}) = \mathcal{A}\delta(\mathbf{x}-\mathbf{y}), \tag{C6}$$

where the last equality follows from Eq. (C1). In the following we consider two examples. First, let us take

$$\mathcal{A} = \frac{1}{4\pi G}\nabla^2, \qquad \mathcal{K}(\mathbf{x}-\mathbf{y}) = -\frac{G}{|\mathbf{x}-\mathbf{y}|}, \tag{C7}$$

then from Eq. (C6), we have

$$\mathcal{K}^{-1}(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi G}\nabla^2\delta(\mathbf{x}-\mathbf{y}). \tag{C8}$$

A less trivial example is that of the operator

$$\mathcal{A} = \exp\left[-\frac{1}{4}\sigma^2\nabla^2\right], \tag{C9}$$

and the kernel

$$\mathcal{K}(\mathbf{x}-\mathbf{y}) = \frac{1}{(\pi\sigma^2)^{3/2}}\exp\left[-\frac{(\mathbf{x}-\mathbf{y})^2}{\sigma^2}\right]. \tag{C10}$$

Then, it can be shown [29] that

$$\mathcal{K}^{-1}(\mathbf{x}-\mathbf{y}) = \mathcal{K}(\mathbf{x}-\mathbf{y})\prod_{k=1}^{3}\sum_{n_k=0}^{\infty} c_{n_k}H_{2n_k}\left(\frac{x_k-y_k}{\sigma}\right), \tag{C11}$$

where H_{2n_k} are the Hermite polynomials of degree $2n_k$, and $c_{n_k} = (-1)^{2n_k}/(2^{n_k}n_k!)$.

Appendix D: Another implementation of the gravitational interaction?

Here, we discuss another possible implementation of the gravitational interaction through a feedback process, which however leads to inconsistencies at the level of the master equation.

For the sake of simplicity, we consider the one dimensional case. We consider one measurement record for each of the N particles of the same form as in Eq. (4). In such a way, once the position of one particle is measured, the other particles receive the same measurement record. A graphical scheme is shown in Fig. 4. After the continuous measurements, which is described by Eq. (2)

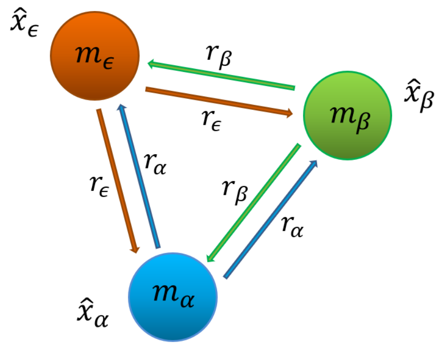

FIG. 4: Graphical representation of the measurement record distribution which leads to the master equation (D2).

with the sum running over all the N particles, one implements the gravitational interaction through the following feedback Hamiltonian

$$\hat{H}_{\mathrm{fb}} = \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \chi_{\alpha\beta} r_\alpha \hat{x}_\beta. \tag{D1}$$

Following the procedure described in Appendix A and Appendix B, we obtain the following master equation

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}_0,\hat{\rho}\right] - \frac{i}{2\hbar}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \chi_{\alpha\beta}\left[\hat{x}_\beta,\{\hat{x}_\alpha,\hat{\rho}\}\right] - \sum_{\alpha=1}^{N}\frac{\gamma_\alpha}{8\hbar^2}\left[\hat{x}_\alpha,\left[\hat{x}_\alpha,\hat{\rho}\right]\right] - \sum_{\substack{\alpha,\beta,\epsilon=1\\ \beta,\epsilon\neq\alpha}}^{N}\frac{\chi_{\alpha\beta}\chi_{\alpha\epsilon}}{2\gamma_\alpha}\left[\hat{x}_\beta,\left[\hat{x}_\epsilon,\hat{\rho}\right]\right]. \tag{D2}$$

This master equation coincides with Eq. (17) only for $N = 2$. In such a case, we obtain the KTM model, where the double commutators involve only the position operator of one particle at the time. For $N > 2$, the last term in Eq. (D2) involves the position operators of different particles. This makes Eq. (D2) physically inconsistent.

In fact, let us consider two bodies of N_1 and N_2 particles respectively, therefore the master equation (D2) involves $N = N_1 + N_2$ particles. However, if we now consider these two bodies as single objects, whose center of mass position operators are $\hat{x}_1$ and $\hat{x}_2$ respectively, after tracing over the relative degrees of freedom of each of the two bodies, we should obtain the KTM master equation for the $N = 2$ objects. However, we get extra contributions proportional to $[\hat{x}_\alpha,[\hat{x}_\beta,\hat{\rho}]]$ with $\alpha \neq \beta$, which do not appear in the two-particle case. This means that composite systems cannot be effectively considered as single particles, as typical in physical theories when the details of the internal dynamics are not relevant.

These problematic terms do not appear in the generalization of the KTM model presented in the main text, nor in that by Altamirano *et al.* [21]. Conversely, in the TD model there are double-commutator terms involving different particles already in the case of two particles. Therefore this inconsistency is not present there.

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